

# Dressing operator approach to Moyal algebraic deformation of selfdual gravity

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## Abstract

Recently Strachan introduced a Moyal algebraic deformation of selfdual gravity, replacing a Poisson bracket of the Plebanski equation by a Moyal bracket. The dressing operator method in soliton theory can be extended to this Moyal algebraic deformation of selfdual gravity. Dressing operators are defined as Laurent series with coefficients in the Moyal (or star product) algebra, and turn out to satisfy a factorization relation similar to the case of the KP and Toda hierarchies. It is a loop algebra of the Moyal algebra (i.e., of a  $W_\infty$  algebra) and an associated loop group that characterize this factorization relation. The nonlinear problem is linearized on this loop group and turns out to be integrable.

*Key words:* dressing operator, Moyal algebra, self-dual gravity,  
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## 1. Introduction

The notion of Moyal algebras [1] is a kind of quantum deformation of Poisson algebras. Since the end of the eighties, there has been renewed interest in these algebras. This is because in two dimensions, they give an explicit realization of the two types of  $W$ -infinity algebras—quantum ( $W_\infty$ ) and classical ( $w_\infty$ ) algebras. It is now widely recognized that  $W$ -infinity algebras of both types are deeply linked with integrability of nonlinear systems.

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A number of integrable systems are now known to be related to  $w_\infty$  algebras. Even within the context of field theory, one can pick out several important examples such as: the dispersionless KP hierarchy [2–4], the  $SU(\infty)$  Toda field theory [5–7], its hierarchy (the dispersionless Toda hierarchy) [8,9], the selfdual vacuum Einstein equation (selfdual gravity) [10–13], etc. In these integrable systems,  $w_\infty$  algebras are realized as the Poisson algebra  $\text{Poisson}(\Sigma)$  or the algebra  $\text{sdiff}(\Sigma)$  of area-preserving diffeomorphisms on a two dimensional surface  $\Sigma$ . Penrose’s twistor theory [14] provides a unified framework for understanding integrability of these systems.

One may naturally ask if these integrable systems of  $w_\infty$  type have any integrable deformation associated with a  $W_\infty$  algebra. The dispersionless KP and Toda hierarchies do have such a  $W_\infty$  analogue, i.e., the ordinary KP and Toda hierarchies, which are of course integrable. A  $W_\infty$  analogue of selfdual gravity was recently proposed by Strachan [15] as a Moyal algebraic deformation of the selfdual vacuum Einstein equation. The problem of proving its integrability, however, remains obscure.

In this paper, we consider this integrability problem by means of soliton theoretical techniques rather than of twistor theory. In soliton theory, the notion of “dressing operators” plays a central role. Our strategy is to construct dressing operators for Strachan’s deformation of selfdual gravity. We will then derive a “factorization relation” that connects those dressing operators with their “initial values” on a two dimensional subspace of space–time. This technique is borrowed from a similar approach to the KP hierarchy [16] and the Toda hierarchy [17], and as in those cases, we can thereby show that the nonlinear problem is converted into a linear problem on an infinite dimensional group.

## 2. Plebanski equation and Moyal algebraic deformation

Let us recall that the Plebanski equation [18]

$$\{\Omega_{,p}, \Omega_{,q}\}_{\text{PB}} \equiv \Omega_{,p\hat{p}}\Omega_{,q\hat{q}} - \Omega_{,p\hat{q}}\Omega_{,q\hat{p}} = 1, \tag{1}$$

where  $\Omega$  is an unknown function (Kähler potential) of suitable space–time coordinates  $(p, q, \hat{p}, \hat{q})$ , gives a local expression of selfdual vacuum Einstein spaces. Strachan’s idea [15] is to replace the Poisson bracket

$$\{F, G\}_{\text{PB}} = \frac{\partial F}{\partial p} \frac{\partial G}{\partial \hat{q}} - \frac{\partial F}{\partial \hat{q}} \frac{\partial G}{\partial p} \tag{2}$$

by the Moyal bracket

$$\{F, G\}_{\text{MB}} = \frac{2}{\hbar} \sinh \left[ \frac{\hbar}{2} \left( \frac{\partial^2}{\partial p \partial \hat{q}'} - \frac{\partial^2}{\partial \hat{q} \partial p'} \right) \right] F(\hat{p}, \hat{q}) G(\hat{p}', \hat{q}') \Big|_{\hat{p}'=\hat{p}, \hat{q}'=\hat{q}}. \tag{3}$$

[This definition is somewhat unusual, but reproduces the ordinary Moyal bracket if one replaces  $\hbar \rightarrow i\hbar$ . This is harmless, because  $\hbar$  in this note is just a formal parameter. This substitution rule  $\hbar \rightarrow i\hbar$  applies to all formulas in the following.] The deformed equation

$$\{\Omega_p, \Omega_q\}_{\text{MB}} = 1 \tag{4}$$

reduces to Eq. (1) in the quasi-classical ( $\hbar \rightarrow 0$ ) limit.

### 3. Twistor functions and Lax formalism

Integrability of selfdual gravity is related to the existence of auxiliary variables, so called “twistor functions” [19] They are functions of both space–time coordinates and a “spectral parameter”  $\lambda$ , and define a correspondence between the space–time and the twistor space. For local analysis of selfdual gravity, it is sufficient to consider four such functions  $\mathcal{U}, \mathcal{V}, \hat{\mathcal{U}}, \hat{\mathcal{V}}$  [10,11]. They have Laurent series expansion

$$\begin{aligned} \mathcal{U} &= \lambda p + \sum_{n=0}^{\infty} u_n \lambda^{-n}, & \mathcal{V} &= \lambda q + \sum_{n=0}^{\infty} v_n \lambda^{-n}, \\ \hat{\mathcal{U}} &= \hat{p} + \sum_{n=1}^{\infty} \hat{u}_n \lambda^n, & \hat{\mathcal{V}} &= \hat{q} + \sum_{n=1}^{\infty} \hat{v}_n \lambda^n, \end{aligned} \tag{5}$$

and satisfy the “Lax equations”

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial p} + \lambda \left\{ \frac{\partial \Omega}{\partial p}, \mathcal{U} \right\}_{\text{PB}} &= 0, & \frac{\partial \mathcal{U}}{\partial q} + \lambda \left\{ \frac{\partial \Omega}{\partial q}, \mathcal{U} \right\}_{\text{PB}} &= 0, \\ \dots \text{ (same equations with } \mathcal{U} \text{ replaced by } \mathcal{V}, \hat{\mathcal{U}}, \hat{\mathcal{V}} \text{) } \dots \end{aligned} \tag{6}$$

and the “canonical Poisson relations”

$$\{\mathcal{U}, \mathcal{V}\}_{\text{PB}} = \{\hat{\mathcal{U}}, \hat{\mathcal{V}}\}_{\text{PB}} = 1. \tag{7}$$

The Plebanski equation, (1), gives the Frobenius integrability condition of these equations. These equations can further be converted into the 2-form equation

$$d\mathcal{U} \wedge d\mathcal{V} = d\hat{\mathcal{U}} \wedge d\hat{\mathcal{V}}, \tag{8}$$

which is a clue for understanding various aspects of integrability of selfdual gravity [10,11,19,20].

It is easy to see how these auxiliary variables can be extended to the deformed equation. Moyal algebraic counterparts of  $\mathcal{U}, \mathcal{V}, \hat{\mathcal{U}}, \hat{\mathcal{V}}$ , say  $U, V, \hat{U}, \hat{V}$ , will be given by Laurent series of the form

$$\begin{aligned}
 U &= \lambda p + \sum_{n=0}^{\infty} u_n \lambda^{-n}, & V &= \lambda q + \sum_{n=0}^{\infty} v_n \lambda^{-n}, \\
 \hat{U} &= \hat{p} + \sum_{n=1}^{\infty} \hat{u}_n \lambda^n, & \hat{V} &= \hat{q} + \sum_{n=1}^{\infty} \hat{v}_n \lambda^n
 \end{aligned}
 \tag{9}$$

with  $\hbar$ -dependent coefficients that have smooth  $\hbar \rightarrow 0$  limit as

$$u_n(\hbar, p, q, \hat{p}, \hat{q}) = u_n^{(0)}(p, q, \hat{p}, \hat{q}) + O(\hbar), \text{ etc } \dots
 \tag{10}$$

The Moyal algebraic version of the Lax equations will be given by

$$\begin{aligned}
 \frac{\partial U}{\partial p} + \lambda \left\{ \frac{\partial \Omega}{\partial p}, U \right\}_{\text{MB}} &= 0, & \frac{\partial U}{\partial q} + \lambda \left\{ \frac{\partial \Omega}{\partial q}, U \right\}_{\text{MB}} &= 0, \\
 \dots \text{ (same equations with } U \text{ replaced by } V, \hat{U}, \hat{V}) \dots
 \end{aligned}
 \tag{11}$$

The canonical Poisson relations will be replaced by the “canonical commutation relations”

$$\{U, V\}_{\text{MB}} = \{\hat{U}, \hat{V}\}_{\text{MB}} = 1.
 \tag{12}$$

One can indeed show that the deformed Plebanski equation, (4), becomes the Frobenius integrability condition of these equations. Obviously, these equations reduce to the previous equations in the quasi-classical ( $\hbar \rightarrow 0$ ) limit.

#### 4. Comparison with KP and dispersionless KP hierarchies

It will be instructive to compare the present situation with the relation between the KP hierarchy and its quasi-classical limit (the dispersionless KP hierarchy) [2–4]. The Lax formalism of the KP hierarchy consists of a set of Lax equations for a canonical conjugate pair of pseudo-differential operators  $L$  and  $M$ ,  $[L, M] = 1$ . In the quasi-classical limit,  $L$  and  $M$  are replaced by a canonical conjugate pair of functions  $\mathcal{L}$  and  $\mathcal{M}$ ,  $\{\mathcal{L}, \mathcal{M}\} = 1$ , on a two dimensional phase space with a Poisson bracket  $\{ , \}$ . They satisfy a set of quasi-classical Lax equations in which the commutator of pseudo-differential operators is replaced by the Poisson bracket. Thus the Lax formalism of the Plebanski equation is almost parallel to the dispersionless KP hierarchy, both being related to a Poisson algebra. We expect a similar correspondence between the Lax formalism of the Moyal algebraic Plebanski equation and the KP hierarchy.

To make this analogy more precise, let us recall that the Moyal bracket can be expressed as a (normalized) commutator

$$\{F, G\}_{\text{MB}} = \frac{2}{\hbar} (F * G - G * F)
 \tag{13}$$

of the star product [1]

$$F * G = \exp \left[ \frac{\hbar}{2} \left( \frac{\partial^2}{\partial \hat{p} \partial \hat{q}'} - \frac{\partial^2}{\partial \hat{q} \partial \hat{p}'} \right) \right] F(\hat{p}, \hat{q}) G(\hat{p}', \hat{q}') \Big|_{\hat{p}' = \hat{p}, \hat{q}' = \hat{q}}. \tag{14}$$

The star product is associative and non-commutative, and in fact coincides with the composition rule of (pseudo)differential operators in “Weyl ordering” (hence gives a realization of  $W_\infty$  algebra). Hoppe et al. [21] use this associative algebraic structure behind the Moyal bracket to study a family of integrable systems related to Moyal algebras. Although those integrable systems are different from ours, this is very suggestive. In our case, basic building blocks of the theory are Laurent series with coefficients in the Moyal algebra, which form a loop algebra of the Moyal algebra. The star product can be extended to this loop algebra as

$$(F\lambda^n) * (G\lambda^m) = F * G\lambda^{n+m} \tag{15}$$

and defines an associative algebraic structure. In the case of the KP hierarchy, the same role is played by pseudo-differential operators, which also form an associative and non-commutative algebra.

### 5. Dressing operators

These observations indicate us what a dressing operator approach to the Moyal algebraic Plebanski equation looks like. The dressing operator of the KP hierarchy is a pseudo-differential operator of the form  $W = 1 + w_1(\partial/\partial x)^{-1} + \dots$ . Dressing operators of the Moyal algebraic Plebanski equation should be Laurent series of  $\lambda$  with coefficients in the Moyal algebra.

More specifically, we need two dressing operators, say  $W$  and  $\hat{W}$  to express the two different types of Lax operators  $(U, V)$  and  $(\hat{U}, \hat{V})$  in a dressing form,

$$\begin{aligned} U &= W * (\hat{p} + p\lambda) * W^{-1}, & V &= W * (\hat{q} + q\lambda) * W^{-1}, \\ \hat{U} &= \hat{W} * \hat{p} * \hat{W}^{-1}, & \hat{V} &= \hat{W} * \hat{q} * \hat{W}^{-1}. \end{aligned} \tag{16}$$

This is rather similar to the Toda hierarchy, and one will therefore infer that these dressing relations can be satisfied by Laurent series of the form

$$\begin{aligned} W &= \sum_{n=0}^{\infty} w_n \lambda^{-n}, & w_0 &\neq 0, \\ \hat{W} &= 1 + \sum_{n=1}^{\infty} \hat{w}_n \lambda^n, & \hat{w}_1 &= -\Omega/\hbar \end{aligned} \tag{17}$$

with coefficients depending on  $\hbar$  and  $(p, q, \hat{p}, \hat{q})$ . This is, however, not yet enough if we take into account the requirement that the coefficients  $u_n$ , etc.

of the Lax operators behave smoothly in the limit of  $\hbar \rightarrow 0$  as shown in (10). This requirement is fulfilled if  $W$  and  $\hat{W}$  are written

$$\begin{aligned} W &= \exp_* \hbar^{-1} A(\hbar, p, q, \hat{p}, \hat{q}, \lambda), & A &= \sum_{n=0}^{\infty} a_n(\hbar, p, q, \hat{p}, \hat{q}) \lambda^{-n}, \\ \hat{W} &= \exp_* \hbar^{-1} \hat{A}(\hbar, p, q, \hat{p}, \hat{q}, \lambda), & \hat{A} &= \sum_{n=1}^{\infty} \hat{a}_n(\hbar, p, q, \hat{p}, \hat{q}) \lambda^n, \end{aligned} \tag{18}$$

with coefficients  $a_n$  and  $\hat{a}_n$  that have smooth limit as  $\hbar \rightarrow 0$ . If we select these dressing operators appropriately, the Lax equations can be converted into the evolution equations

$$\begin{aligned} \hbar \frac{\partial W}{\partial p} &= -\lambda \frac{\partial \Omega}{\partial p} * W + W * \hat{q} \lambda, \\ \hbar \frac{\partial W}{\partial q} &= -\lambda \frac{\partial \Omega}{\partial q} * W - W * \hat{p} \lambda, \\ \hbar \frac{\partial \hat{W}}{\partial p} &= \left( -\lambda \frac{\partial \Omega}{\partial p} - \frac{1}{2} q \lambda^2 \right) * \hat{W}, \\ \hbar \frac{\partial \hat{W}}{\partial q} &= \left( -\lambda \frac{\partial \Omega}{\partial q} + \frac{1}{2} p \lambda^2 \right) * \hat{W} \end{aligned} \tag{19}$$

of the dressing operators. This is quite parallel to the dressing operator formalism of the KP and Toda hierarchies, apart from the strange extra terms  $q\lambda^2/2$  and  $p\lambda^2/2$ . These extra terms originate in non-commutativity of the two flows ( $p$ -flow and  $q$ -flow) as we shall see later.

### 6. Factorization relation in $W_\infty$ loop group

We can now apply the factorization technique for the KP and Toda hierarchies [16,17] to our problem. Consequently, it turns out that the dressing operators and their “initial values”  $W_{\text{in}} = W|_{p=q=0}$  and  $\hat{W}_{\text{in}} = \hat{W}|_{p=q=0}$  are connected by the factorization relation

$$W^{-1} * \hat{W} = e(p, q) * W_{\text{in}}^{-1} * \hat{W}_{\text{in}}. \tag{20}$$

The boost operator  $e(p, q)$  of time evolution is given by

$$e(p, q) = \exp_* [\hbar^{-1} (-p\hat{q}\lambda + q\hat{p}\lambda)], \tag{21}$$

where  $\exp_*$  is the star exponential,

$$\exp_* F = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} F * \dots * F \quad (n\text{-fold product}). \tag{22}$$

This factorization relation may be understood as an integrated form of Eq. (19). If, conversely, one can solve this relation for a given initial data,  $W$  and  $\hat{W}$  automatically satisfy Eq. (19) hence give rise to a solution of the Moyal algebraic Plebanski equation.

Solvability of this factorization problem is ensured, at least in a neighborhood of  $(p, q) = (0, 0)$ , by the following reasoning. The set of Laurent series of the form  $\hbar^{-1}A + \hbar^{-1}\hat{A}$ , where  $A$  and  $\hat{A}$  are as in Eq. (18), is closed under the star product commutator, therefore gives a Lie algebra  $\mathcal{G}$  with a natural direct sum decomposition into two Lie subalgebras,

$$\mathcal{G} = \mathcal{G}_{\leq 0} \oplus \mathcal{G}_{\geq 1}, \quad \hbar^{-1}A \in \mathcal{G}_{\leq 0}, \quad \hbar^{-1}\hat{A} \in \mathcal{G}_{\geq 1}. \tag{23}$$

This induces a decomposition at the group level,

$$\exp \mathcal{G} = \exp \mathcal{G}_{\leq 0} \cdot \exp \mathcal{G}_{\geq 1}, \tag{24}$$

at least in a neighborhood of the identity element. Thus, given initial data  $W_{\text{in}} \in \exp \mathcal{G}_{\leq 0}$  and  $\hat{W}_{\text{in}} \in \exp \mathcal{G}_{\geq 1}$ , one can find two factors  $W \in \exp \mathcal{G}_{\leq 0}$  and  $\hat{W} \in \exp \mathcal{G}_{\geq 1}$  that satisfy the factorization relation.

The Lie algebra  $\mathcal{G}$  is a kind of loop algebra of the Moyal algebra (hence of a  $W_\infty$  algebra). This gives a quantum deformation of a similar loop algebra of the Poisson algebra (i.e., of a  $w_\infty$  algebra) in the ordinary Plebanski equation [10,11]. As we have seen, it is this loop algebra rather than the Moyal algebra itself that characterizes diverse hidden structures of the Moyal algebraic Plebanski equation.

Eq. (19) can now be interpreted as integrable flows on the direct product group  $\exp \mathcal{G}_{\leq 0} \times \exp \mathcal{G}_{\geq 1}$ . As Mulase described impressively in the case of the KP hierarchy [16], the factorization relation now works as a machinery that links the nonlinear world (the left hand side) with the linear world (the right hand side) in which the flows are “linearized” into the left action of  $e(p, q)$ . Furthermore, the two flows are non-commutative, because the two terms in the star exponential of  $e(p, q)$  do not commute with respect to the star product. To be more specific,  $e(p, q)$  can be factorized as

$$\begin{aligned} \exp_* [\hbar^{-1}(-p\hat{q}\lambda + q\hat{p}\lambda)] &= \exp[\hbar^{-1}(-p\hat{q}\lambda - \frac{1}{2}pq\lambda^2)] * \exp[\hbar^{-1}q\hat{p}\lambda] \\ &= \exp[\hbar^{-1}(q\hat{p}\lambda + \frac{1}{2}pq\lambda^2)] * \exp[-\hbar^{-1}p\hat{q}\lambda], \end{aligned} \tag{25}$$

and thereby satisfies an unusual differentiation formula:

$$\begin{aligned} \hbar \frac{\partial e(p, q)}{\partial p} &= (-\hat{q}\lambda - \frac{1}{2}q\lambda^2) * e(p, q), \\ \hbar \frac{\partial e(p, q)}{\partial q} &= (\hat{p}\lambda + \frac{1}{2}p\lambda^2) * e(p, q). \end{aligned} \tag{26}$$

This is the origin of the extra terms  $p\lambda^2/2$  and  $q\lambda^2/2$  in Eq. (19).

## 7. Conclusion

We have thus seen that the dressing operator method in soliton theory can be extended to the Moyal algebraic Plebanski equation. Dressing operators are defined as Laurent series with coefficients in the Moyal (or star product) algebra, and turn out to satisfy a factorization relation similar to the case of the KP and Toda hierarchies. It is a loop algebra of the Moyal algebra (i.e., of a  $W_\infty$  algebra) and an associated loop group that characterize this factorization relation. The nonlinear problem is linearized on this loop group and turns out to be integrable.

We add a few comments.

1. A hierarchy of higher flows can be constructed as in the case of the KP and Toda hierarchies. One set of such flows are parametrized by two series of variables  $p_n, q_n, n = 1, 2, \dots$  ( $p_1 = p, q_1 = q$ ) and generated by the star exponential

$$e(p_1, p_2, \dots, q_1, q_2, \dots) = \exp_* \left[ \hbar^{-1} \left( - \sum_{n=1}^{\infty} p_n \hat{q} \lambda^n + \sum_{n=1}^{\infty} q_n \hat{p} \lambda^n \right) \right] \quad (27)$$

instead of the previous  $e(p, q)$ . Another set of flows with variables  $\hat{p}_n, \hat{q}_n, n = 1, 2, \dots$ , which resemble the “negative flows” of the Toda hierarchy, are generated by inserting a similar star exponential (but replacing  $\lambda^n \rightarrow \lambda^{-n}$ ) to the right side of  $W_{\text{in}}^{-1} * \hat{W}_{\text{in}}$  in Eq. (20). The results of this paper can be extended straightforward to these higher flows.

2. The factorization relation can also be used to construct a large set of symmetries ( $W_\infty$  symmetries). These symmetries are generated by the action of a star product loop group element from the left or right side of  $W_{\text{in}}^{-1} * \hat{W}_{\text{in}}$ . The aforementioned hierarchy of higher flows is just a subset of these symmetries. In the quasi-classical ( $\hbar \rightarrow 0$ ) limit, these symmetries reproduce  $w_\infty$  symmetries of the Plebanski equation [10,11].

3. The Plebanski equation is interpreted as the equation of motion of a physical state in  $N = 2$  string theory [22]. Can we find a similar interpretation of the Moyal algebraic deformation?

4. It is also quite straightforward to extend our results to higher ( $2k, k = 1, 2, \dots$ ) dimensional Moyal algebras. This leads to a Moyal algebraic deformation of hyper-Kähler geometry. Hyper-Kähler manifolds are known to give target spaces of supersymmetric sigma models [23]. Can we find a similar application of the Moyal algebraic deformation of hyper-Kähler geometry?

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